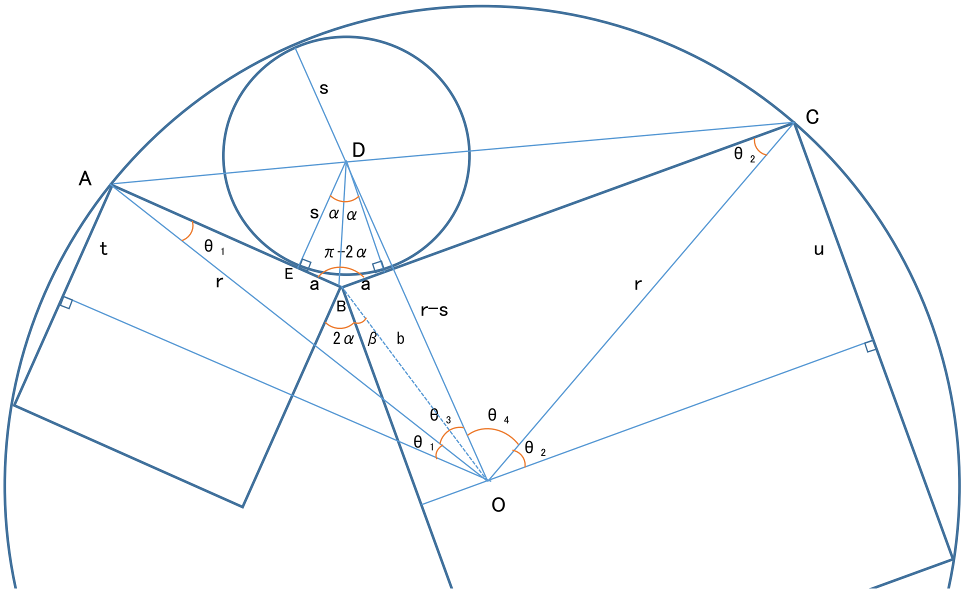
令和5年2月の問題ーNo.3



小方、大方の一辺の長さをそれぞれ 2t, 2u とし、小円、大円の半径をそれぞれ s, r とする。
また、線分EB = a、線分OB = b とする。

$$\cos(\theta_1) = \frac{\sqrt{r^2 - t^2}}{r} \qquad \cos(\theta_2) = \frac{\sqrt{r^2 - u^2}}{r}$$

△OABより、 $b^2 = r^2 + (2t)^2 - 2r(2t)\cdot\cos(\theta_1) = r^2 + 4t^2 - 4t\cdot\sqrt{r^2 - t^2}$

△OBCより、 $b^2 = r^2 + (2u)^2 - 2r(2u)\cdot\cos(\theta_2) = r^2 + 4u^2 - 4u\cdot\sqrt{r^2 - u^2}$

$r^2 + 4t^2 - 4t\cdot\sqrt{r^2 - t^2} = r^2 + 4u^2 - 4u\cdot\sqrt{r^2 - u^2}$

$u\cdot\sqrt{r^2 - u^2} - t\cdot\sqrt{r^2 - t^2} = u^2 - t^2$

$2ut\cdot\sqrt{(r^2 - u^2)(r^2 - t^2)} = (u^2 + t^2)r^2 - 2(u^4 + t^4 - u^2t^2)$

$(u^2 - t^2)^2\left[r^4 - 4(u^2 + t^2)r^2 + 4(u^4 + t^4)\right] = 0$

$u > t$ より、 $r^4 - 4(u^2 + t^2)r^2 + 4(u^2 + t^2)^2 - 8u^2t^2 = 0$

$\left[r^2 - 2(u^2 + t^2)\right]^2 = 8u^2t^2$

$r^2 = 2(u^2 + t^2 + \sqrt{2}ut)$ —— ① $r^2 = 2(u^2 + t^2 - \sqrt{2}ut)$ —— ①'

$\cos(2\alpha + \beta) = \frac{t}{b}$ $\cos(2\alpha + \beta) = \cos(2\alpha)\cdot\cos(\beta) - \sin(2\alpha)\cdot\sin(\beta) = \cos(2\alpha)\cdot\frac{u}{b} - \sin(2\alpha)\cdot\frac{2u - \sqrt{r^2 - u^2}}{b} = \frac{t}{b}$

$u\cdot\cos(2\alpha) - (2u - \sqrt{r^2 - u^2})\cdot\sin(2\alpha) = t$

$$\tan(\alpha) = \frac{a}{s} \qquad \tan(2\alpha) = \frac{2\cdot\tan(\alpha)}{1 - (\tan(\alpha))^2} = \frac{2sa}{s^2 - a^2}$$

$$\cos(2\alpha) = \frac{1}{\sqrt{1 + (\tan(2\alpha))^2}} = \frac{s^2 - a^2}{s^2 + a^2} \qquad \sin(2\alpha) = \sqrt{1 - (\cos(2\alpha))^2} = \frac{2sa}{s^2 + a^2}$$

$$u\cdot\frac{s^2 - a^2}{s^2 + a^2} - (2u - \sqrt{r^2 - u^2})\cdot\left(\frac{2sa}{s^2 + a^2}\right) = t$$

式①より、 $r^2 - u^2 = u^2 + 2t^2 + 2\sqrt{2}ut = (u + \sqrt{2}t)^2$

$u\cdot(s^2 - a^2) - [2u - (u + \sqrt{2}t)]\cdot 2sa = t(s^2 + a^2)$

$(u + t)a^2 + 2(u - \sqrt{2}t)sa - (u - t)s^2 = 0$

(i) $a = \frac{-(u - \sqrt{2}t)s + (\sqrt{2}u - t)s}{u + t} = \frac{(\sqrt{2} - 1)(u + t)}{u + t}s = (\sqrt{2} - 1)s$

(ii) $a = \frac{-(u - \sqrt{2}t)s - (\sqrt{2}u - t)s}{u + t} = \frac{-(\sqrt{2} + 1)(u - t)}{u + t}s < 0$ なので除外する。

$a = (\sqrt{2} - 1)s$ —— ②

$\tan(\alpha) = \frac{a}{s} = \sqrt{2} - 1$ $\tan(2\alpha) = \frac{2sa}{s^2 - a^2} = \frac{2s(\sqrt{2} - 1)s}{s^2 - (\sqrt{2} - 1)^2s^2} = 1$ $\cos(2\alpha) = \frac{1}{\sqrt{1 + (\tan(2\alpha))^2}} = \frac{1}{\sqrt{2}}$

△ABC、△OACより、 $(2t)^2+(2u)^2-2(2t)(2u)\cdot\cos(\pi-2\alpha)=r^2+r^2-2r^2\cdot\cos(\theta_3+\theta_4)$

$$\cos(\pi-2\alpha)=\cos(\pi)\cdot\cos(2\alpha)+\sin(\pi)\cdot\sin(2\alpha)=-\cos(2\alpha)=\frac{-1}{\sqrt{2}}$$

$4\cdot\left(u^2+t^2+\sqrt{2}\cdot ut\right)=2r^2\left(1-\cos(\theta_3+\theta_4)\right)$

式①より、 $2r^2=2r^2\left(1-\cos(\theta_3+\theta_4)\right)$

$\cos(\theta_3+\theta_4)=0$

$\theta_3+\theta_4=\frac{\pi}{2}$ ―― ③

△OADより、 $s^2+(2t-a)^2=r^2+(r-s)^2-2r(r-s)\cdot\cos(\theta_3)$ $\cos(\theta_3)=1-\frac{(2t-a)^2}{2r(r-s)}$

△ODCより、 $s^2+(2u-a)^2=r^2+(r-s)^2-2r(r-s)\cdot\cos(\theta_4)$ $\cos(\theta_4)=1-\frac{(2u-a)^2}{2r(r-s)}$

$\sin(\theta_3)=\sqrt{1-(\cos(\theta_3))^2}=\frac{(2t-a)\cdot\sqrt{4r(r-s)-(2t-a)^2}}{2r(r-s)}$

式③より、 $\theta_4=\frac{\pi}{2}-\theta_3$ $\cos(\theta_4)=\cos\left(\frac{\pi}{2}-\theta_3\right)=\cos\left(\frac{\pi}{2}\right)\cdot\cos(\theta_3)+\sin\left(\frac{\pi}{2}\right)\cdot\sin(\theta_3)=\sin(\theta_3)$

$1-\frac{(2u-a)^2}{2r(r-s)}=\frac{(2t-a)\cdot\sqrt{4r(r-s)-(2t-a)^2}}{2r(r-s)}$

$2r(r-s)-(2u-a)^2=(2t-a)\cdot\sqrt{4r(r-s)-(2t-a)^2}$

$4r^2\cdot(r-s)^2-4r(r-s)\cdot(2u-a)^2+(2u-a)^4=(2t-a)^2\cdot\left[4r(r-s)-(2t-a)^2\right]$

$(2r(r-s))^2-2\left[(2u-a)^2+(2t-a)^2\right](2r(r-s))+(2u-a)^4+(2t-a)^4=0$

(i) $2r(r-s)=(2u-a)^2+(2t-a)^2+\sqrt{2}(2u-a)(2t-a)=4\left(u^2+t^2+\sqrt{2}ut\right)-2(u+t)(2+\sqrt{2})a+(2+\sqrt{2})a^2$

式①より、 $2r^2-2rs=2r^2-2\cdot(u+t)\cdot(2+\sqrt{2})\cdot a+(2+\sqrt{2})\cdot a^2$

$-2rs=(2+\sqrt{2})a\left[a-2(u+t)\right]$

式②より、 $2r=\frac{(2+\sqrt{2})\cdot(\sqrt{2}-1)s\left[2\cdot(u+t)-(\sqrt{2}-1)s\right]}{s}=\sqrt{2}\left[(2u+2t)-\frac{\sqrt{2}-1}{2}(2s)\right]$

(ii) $2r(r-s)=(2u-a)^2+(2t-a)^2-\sqrt{2}(2u-a)(2t-a)=4\left(u^2+t^2-\sqrt{2}ut\right)-2(u+t)(2-\sqrt{2})a+(2-\sqrt{2})a^2$

式①'より、$2r^2-2rs=2r^2-2\cdot(u+t)\cdot(2-\sqrt{2})\cdot a+(2-\sqrt{2})\cdot a^2$

$-2rs=(2-\sqrt{2})a\left[a-2(u+t)\right]$

式②より、 $2r=\frac{(2-\sqrt{2})\cdot(\sqrt{2}-1)s\left[2\cdot(u+t)-(\sqrt{2}-1)s\right]}{s}=(3\sqrt{2}-4)\left[(2u+2t)-\frac{\sqrt{2}-1}{2}(2s)\right]<u+t$ なので除外する。

大円の直径 = $\sqrt{2}$ ×（小方の一辺の長さと大方の一辺の長さの和）− $\frac{\sqrt{2}-1}{\sqrt{2}}$ ×（小円の直径）