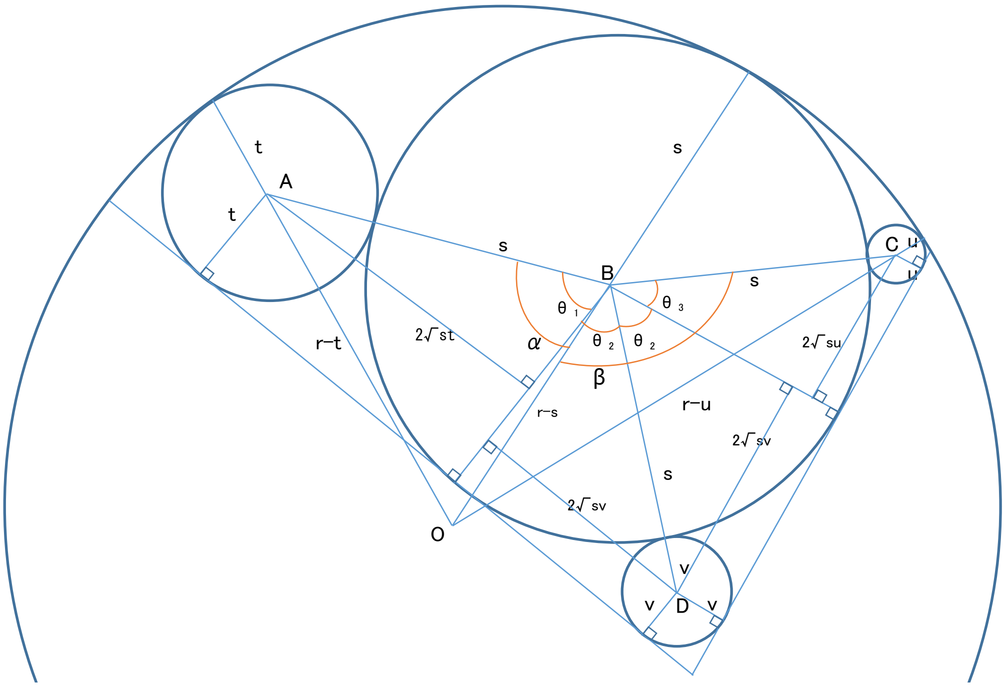
令和5年3月の問題ーNo.1



外円、甲円、乙円、丙円、丁円の半径をそれぞれ r, s, t, u, v とする。

$$\cos(\theta_1) = \frac{s-t}{s+t} \qquad \cos(\theta_2) = \frac{s-v}{s+v} \qquad \cos(\theta_3) = \frac{s-u}{s+u}$$

$$\sin(\theta_1) = \frac{2\sqrt{st}}{s+t} \qquad \sin(\theta_2) = \frac{2\sqrt{sv}}{s+v} \qquad \sin(\theta_3) = \frac{2\sqrt{su}}{s+u}$$

△OABより、 $(r-t)^2 = (r-s)^2 + (s+t)^2 - 2(r-s)(s+t)\cdot\cos(\alpha)$

$$\cos(\alpha) = \frac{2rt}{(r-s)(s+t)} - 1 \qquad \sin(\alpha) = \sqrt{1-\left(\cos(\alpha)\right)^2} = \frac{2\sqrt{rst(r-s-t)}}{(r-s)(s+t)}$$

△OBCより、 $(r-u)^2 = (r-s)^2 + (s+u)^2 - 2(r-s)(s+u)\cdot\cos(\beta)$

$$\cos(\beta) = \frac{2ru}{(r-s)(s+u)} - 1 \qquad \sin(\beta) = \frac{2\sqrt{rsu(r-s-u)}}{(r-s)(s+u)}$$

$$\cos(\alpha+\beta) = \cos(\alpha)\cdot\cos(\beta) - \sin(\alpha)\cdot\sin(\beta) = \left[\frac{2rt}{(r-s)(s+t)} - 1\right]\left[\frac{2ru}{(r-s)(s+u)} - 1\right] - \frac{2\sqrt{rst(r-s-t)}}{(r-s)(s+t)}\cdot\frac{2\sqrt{rsu(r-s-u)}}{(r-s)(s+u)}$$

$$\cos(\alpha+\beta) = 1 - \frac{2rs\left(\sqrt{u(r-s-t)} + \sqrt{t(r-s-u)}\right)^2}{(r-s)^2(s+t)(s+u)}$$

$$\alpha+\beta = \theta_1+\theta_3+2\theta_2 \qquad \cos(\alpha+\beta) = \cos(\theta_1+\theta_3+2\theta_2)$$

$$\cos(\theta_1+\theta_3) = \cos(\theta_1)\cdot\cos(\theta_3) - \sin(\theta_1)\cdot\sin(\theta_3) = \frac{(s-t)\cdot(s-u) - 4\cdot s\cdot\sqrt{ut}}{(s+t)(s+u)}$$

$$\sin(\theta_1+\theta_3) = \sqrt{1-\left(\cos(\theta_1+\theta_3)\right)^2} = \frac{2\cdot\sqrt{s}\cdot(\sqrt{t}+\sqrt{u})\cdot(s-\sqrt{tu})}{(s+t)(s+u)}$$

$$\cos(2\theta_2) = \left(\cos(\theta_2)\right)^2 - \left(\sin(\theta_2)\right)^2 = \frac{(s-v)^2 - 4sv}{(s+v)^2}$$

$$\sin(2\theta_2) = 2\cdot\sin(\theta_2)\cdot\cos(\theta_2) = \frac{4\cdot\sqrt{sv}\cdot(s-v)}{(s+v)^2}$$

$$\cos(\theta_1+\theta_3+2\theta_2) = \cos(\theta_1+\theta_3)\cdot\cos(2\theta_2) - \sin(\theta_1+\theta_3)\cdot\sin(2\theta_2)$$

$$= \frac{(s-t)\cdot(s-u) - 4s\cdot\sqrt{ut}}{(s+t)(s+u)}\cdot\frac{(s-v)^2 - 4sv}{(s+v)^2} - \frac{2\cdot\sqrt{s}\cdot(\sqrt{t}+\sqrt{u})\cdot(s-\sqrt{tu})}{(s+t)(s+u)}\cdot\frac{4\cdot\sqrt{sv}\cdot(s-v)}{(s+v)^2}$$

$$= \frac{\left[(s-t)\cdot(s-u) - 4s\cdot\sqrt{ut}\right]\cdot\left[(s-v)^2 - 4sv\right] - \left(8s\cdot\sqrt{v}\right)\cdot(\sqrt{t}+\sqrt{u})\cdot(s-\sqrt{tu})\cdot(s-v)}{(s+t)(s+u)(s+v)^2}$$

$$1 - \frac{2rs\left(\sqrt{u(r-s-t)} + \sqrt{t(r-s-u)}\right)^2}{(r-s)^2(s+t)(s+u)} = \frac{\left[(s-t)\cdot(s-u) - 4s\cdot\sqrt{ut}\right]\cdot\left[(s-v)^2 - 4sv\right] - \left(8s\cdot\sqrt{v}\right)\cdot(\sqrt{t}+\sqrt{u})\cdot(s-\sqrt{tu})\cdot(s-v)}{(s+t)(s+u)(s+v)^2}$$

$$(r-s)^2 \cdot \left[\left[\begin{array}{l}(s+t)\cdot(s+u)\cdot(s+v)^2-(s-t)\cdot(s-u)\cdot(s-v)^2+4\cdot sv(s-t)\cdot(s-u)\ \dots \\ +4\cdot s\cdot\sqrt{ut}\cdot(s-v)^2-16\cdot s^2\cdot v\cdot\sqrt{ut}+8s\cdot\sqrt{v}(\sqrt{t}+\sqrt{u})(s-\sqrt{tu})(s-v)\end{array}\right]\right]=2rs(s+v)^2\left(\sqrt{u(r-s-t)}+\sqrt{t(r-s-u)}\right)^2$$

$$(r-s)^2\left[2\sqrt{v}(s-\sqrt{tu})+(\sqrt{t}+\sqrt{u})(s-v)\right]^2=r(s+v)^2\left(\sqrt{u(r-s-t)}+\sqrt{t(r-s-u)}\right)^2$$

$$(r-s)\left[2\sqrt{v}(s-\sqrt{tu})+(\sqrt{t}+\sqrt{u})(s-v)\right]=\sqrt{r}(s+v)\left(\sqrt{u(r-s-t)}+\sqrt{t(r-s-u)}\right)$$

$$\left[\sqrt{r}\left(\sqrt{u(r-s-t)}+\sqrt{t(r-s-u)}\right)+(r-s)(\sqrt{t}+\sqrt{u})\right]v-2(r-s)(s-\sqrt{tu})\cdot\sqrt{v}+\left[\sqrt{r}\left(\sqrt{u(r-s-t)}+\sqrt{t(r-s-u)}\right)-(r-s)(\sqrt{t}+\sqrt{u})\right]s=0$$

(i) $$\sqrt{v}=\frac{(r-s)\cdot(s-\sqrt{tu})+\sqrt{(r-s)^2\cdot(s+t)\cdot(s+u)-rs\left(\sqrt{u(r-s-t)}+\sqrt{t(r-s-u)}\right)^2}}{\sqrt{r}\left(\sqrt{u(r-s-t)}+\sqrt{t(r-s-u)}\right)+(r-s)(\sqrt{t}+\sqrt{u})}$$

(ii) $$\sqrt{v}=\frac{(r-s)\cdot(s-\sqrt{tu})-\sqrt{(r-s)^2\cdot(s+t)\cdot(s+u)-rs\left(\sqrt{u(r-s-t)}+\sqrt{t(r-s-u)}\right)^2}}{\sqrt{r}\left(\sqrt{u(r-s-t)}+\sqrt{t(r-s-u)}\right)+(r-s)(\sqrt{t}+\sqrt{u})}$$

$r=\frac{225}{2}$　$s=\frac{113}{2}$　$t=\frac{48}{2}$　$u=\frac{12}{2}$　を代入すると、

(i) $\sqrt{v}=\frac{3\sqrt{6}}{2}$　$2v=2\left(\frac{3\sqrt{6}}{2}\right)^2=27$

(ii) $\sqrt{v}=\frac{113\sqrt{6}}{162}$　$2v=2\left(\frac{113\sqrt{6}}{162}\right)^2$ ≒5.83859 ＜ u なので除外する。

∴丁円の直径＝27寸