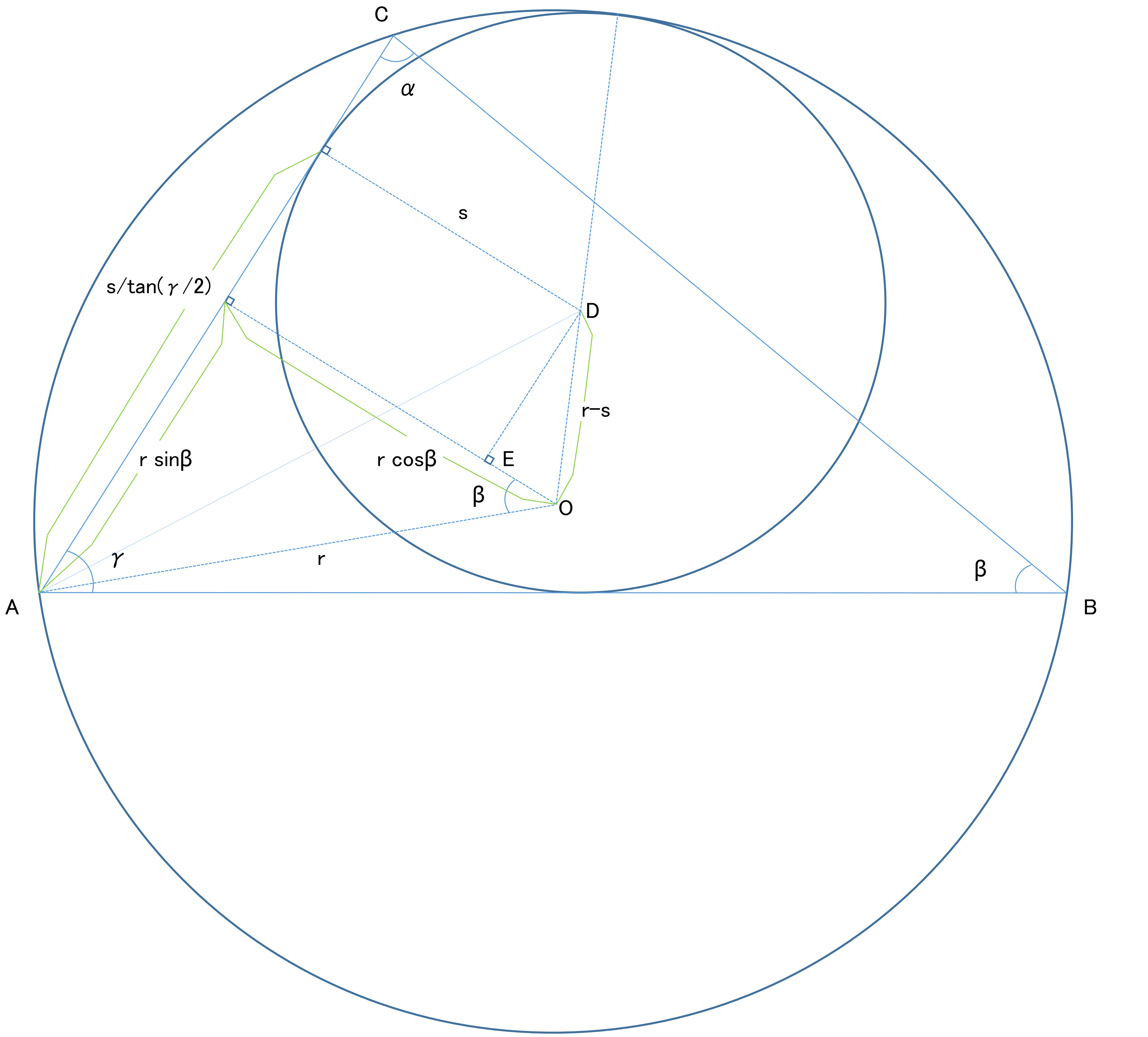

外円の半径を r，乙円の半径を s とし、△ABCの３つの角をα，β，γとする。

$$\alpha+\beta+\gamma=\pi \qquad \sin(\alpha)=\sin(\beta+\gamma) \qquad \sin(\beta)=\sin(\alpha+\gamma) \qquad \sin(\gamma)=\sin(\alpha+\beta)$$

△DEOより、 $$\left(\frac{s}{\tan\left(\frac{\gamma}{2}\right)}-r\cdot\sin(\beta)\right)^2+\left(r\cdot\cos(\beta)-s\right)^2=(r-s)^2$$

$$\frac{s}{\left(\tan\left(\frac{\gamma}{2}\right)\right)^2}=2r\left(\frac{\sin(\beta)}{\tan\left(\frac{\gamma}{2}\right)}+\cos(\beta)-1\right)$$

$$\tan(\gamma)=\frac{2\cdot\tan\left(\frac{\gamma}{2}\right)}{1-\left(\tan\left(\frac{\gamma}{2}\right)\right)^2}$$ $\tan\left(\frac{\gamma}{2}\right)>0$ より、 $\tan\left(\frac{\gamma}{2}\right)=\frac{1-\cos(\gamma)}{\sin(\gamma)}$ $\qquad \frac{1}{\tan\left(\frac{\gamma}{2}\right)}=\frac{1+\cos(\gamma)}{\sin(\gamma)}$

$$(1+\cos(\gamma))^2s=2r\cdot\sin(\gamma)\cdot\left[(1+\cos(\gamma))\cdot\sin(\beta)+(\cos(\beta)-1)\cdot\sin(\gamma)\right]=2r\cdot\sin(\gamma)\cdot(\sin(\beta)-\sin(\gamma)+\sin(\beta)\cdot\cos(\gamma)+\cos(\beta)\cdot\sin(\gamma))$$
$$=2r\cdot\sin(\gamma)\cdot(\sin(\beta)-\sin(\gamma)+\sin(\beta+\gamma))=2r\cdot\sin(\gamma)\cdot(\sin(\alpha)+\sin(\beta)-\sin(\gamma))$$

$$2r=\frac{(1+\cos(\gamma))^2s}{\sin(\gamma)\cdot(\sin(\alpha)+\sin(\beta)-\sin(\gamma))} \qquad ---①$$

甲円の半径を t とし、同様に解くと、 $\left(\dfrac{t}{\tan\left(\dfrac{\alpha}{2}\right)} - r\cdot\sin(\gamma)\right)^2 + \left(t - r\cdot\cos(\gamma)\right)^2 = (r-t)^2$

$$2r = \frac{\left(1+\cos(\alpha)\right)^2 t}{\sin(\alpha)\cdot\left(-\sin(\alpha)+\sin(\beta)+\sin(\gamma)\right)} \quad ---②$$

丙円の半径を u とし、同様に解くと、 $\left(\dfrac{u}{\tan\left(\dfrac{\beta}{2}\right)} - r\cdot\sin(\alpha)\right)^2 + \left(u - r\cdot\cos(\alpha)\right)^2 = (r-u)^2$

$$2r = \frac{\left(1+\cos(\beta)\right)^2 u}{\sin(\beta)\cdot\left(\sin(\alpha)-\sin(\beta)+\sin(\gamma)\right)} \quad ---③$$

$\sin(\gamma) = \sin(\alpha+\beta) = \sin(\alpha)\cdot\cos(\beta) + \cos(\alpha)\cdot\sin(\beta)$

式②、③より、 $\left(1+\cos(\alpha)\right)^2\cdot\sin(\beta)\cdot\left(\sin(\alpha)-\sin(\beta)+\sin(\gamma)\right)t = \left(1+\cos(\beta)\right)^2\cdot\sin(\alpha)\cdot\left(-\sin(\alpha)+\sin(\beta)+\sin(\gamma)\right)u$

左辺 $= \left(1+\cos(\alpha)\right)^2\cdot\sin(\beta)\cdot\left[\sin(\alpha)\cdot\left(1+\cos(\beta)\right) - \sin(\beta)\cdot\left(1-\cos(\alpha)\right)\right]t$

$= \left(1+\cos(\alpha)\right)\cdot\sin(\alpha)\cdot\sin(\beta)\cdot\left[\left(1+\cos(\alpha)\right)\left(1+\cos(\beta)\right) - \sin(\alpha)\cdot\sin(\beta)\right]t$

右辺 $= \left(1+\cos(\beta)\right)^2\cdot\sin(\alpha)\cdot\left[-\sin(\alpha)\cdot\left(1-\cos(\beta)\right) + \sin(\beta)\cdot\left(1+\cos(\alpha)\right)\right]u$

$= \left(1+\cos(\beta)\right)\cdot\sin(\alpha)\cdot\sin(\beta)\cdot\left[\left(1+\cos(\alpha)\right)\left(1+\cos(\beta)\right) - \sin(\alpha)\cdot\sin(\beta)\right]u$

$\left(1+\cos(\alpha)\right)t = \left(1+\cos(\beta)\right)u$

$$\cos(\beta) = \frac{\left(1+\cos(\alpha)\right)t}{u} - 1 \quad ---④$$

$\sin(\beta) = \sin(\alpha+\gamma) = \sin(\alpha)\cdot\cos(\gamma) + \cos(\alpha)\cdot\sin(\gamma)$

式①、②より、 $\left(1+\cos(\alpha)\right)^2\cdot\sin(\gamma)\cdot\left(\sin(\alpha)+\sin(\beta)-\sin(\gamma)\right)t = \left(1+\cos(\gamma)\right)^2\cdot\sin(\alpha)\cdot\left(-\sin(\alpha)+\sin(\beta)+\sin(\gamma)\right)s$

$\left(1+\cos(\alpha)\right)t = \left(1+\cos(\gamma)\right)s$

$$\cos(\gamma) = \frac{\left(1+\cos(\alpha)\right)t}{s} - 1 \quad ---⑤$$

$\left(1+\cos(\alpha)\right)t = A$ とおくと、 $\cos(\alpha) = \dfrac{A}{t} - 1$ 　式④、⑤より、 $\cos(\beta) = \dfrac{A}{u} - 1$ 　 $\cos(\gamma) = \dfrac{A}{s} - 1$

$$\cos(\beta+\gamma) = \cos(\beta)\cdot\cos(\gamma) - \sin(\beta)\cdot\sin(\gamma) = \left(\frac{A}{u}-1\right)\cdot\left(\frac{A}{s}-1\right) - \sqrt{\left[1-\left(\frac{A}{u}-1\right)^2\right]\cdot\left[1-\left(\frac{A}{s}-1\right)^2\right]}$$

$$= \frac{A^2}{us} - \left(\frac{1}{u}+\frac{1}{s}\right)A - \frac{A}{us}\cdot\sqrt{(2u-A)\cdot(2s-A)} + 1 = -\cos(\alpha)$$

$\dfrac{A}{us} - \left(\dfrac{1}{u}+\dfrac{1}{s}\right) - \dfrac{1}{us}\cdot\sqrt{(2u-A)(2s-A)} + \dfrac{1}{t} = 0$ 　 $t\cdot\sqrt{(2u-A)\cdot(2s-A)} = tA - [(u+s)t - us]$

$$A = 2t - \frac{[(u+s)t-us]^2}{2ust}$$

$\sin(\alpha) = \sqrt{1-\left(\cos(\alpha)\right)^2} = \sqrt{1-\left(\dfrac{A}{t}-1\right)^2} = \dfrac{\sqrt{A(2t-A)}}{t}$ 　 $2t - A = \dfrac{[(u+s)t-us]^2}{2ust}$

$\sin(\beta) = \dfrac{\sqrt{A(2u-A)}}{u}$ 　 $2u - A = \dfrac{[(u+s)t-us]^2}{2ust} - 2(t-u) = \dfrac{[us-(s-u)t]^2}{2ust}$

$\sin(\gamma) = \dfrac{\sqrt{A(2s-A)}}{s}$ 　 $2s - A = \dfrac{[(u+s)t-us]^2}{2ust} - 2(t-s) = \dfrac{[us+(s-u)t]^2}{2ust}$

式②より、 $2r = \dfrac{A^2}{t\cdot\sin(\alpha)\left(-\sin(\alpha)+\sin(\beta)+\sin(\gamma)\right)} = \dfrac{A}{\sqrt{2t-A}\left(\dfrac{-\sqrt{2t-A}}{t}+\dfrac{\sqrt{2u-A}}{u}+\dfrac{\sqrt{2s-A}}{s}\right)}$

$$2r = \frac{2t - \dfrac{[(u+s)t-us]^2}{2ust}}{\dfrac{(u+s)t-us}{\sqrt{2ust}}\left[-\left[\dfrac{(u+s)t-us}{t\cdot\sqrt{2ust}}\right] + \dfrac{us-(s-u)t}{u\cdot\sqrt{2ust}} + \dfrac{us+(s-u)t}{s\cdot\sqrt{2ust}}\right]} = \frac{ust\left[4ust^2 - [(u+s)t-us]^2\right]}{[(u+s)t-us]\left[u^2s^2-(s-u)^2t^2\right]}$$

$s = \dfrac{20}{2}$ 　 $t = \dfrac{30}{2}$ 　 $u = \dfrac{15}{2}$ 　を代入すると、 $2r = 46$

∴外円の直径 = 46寸