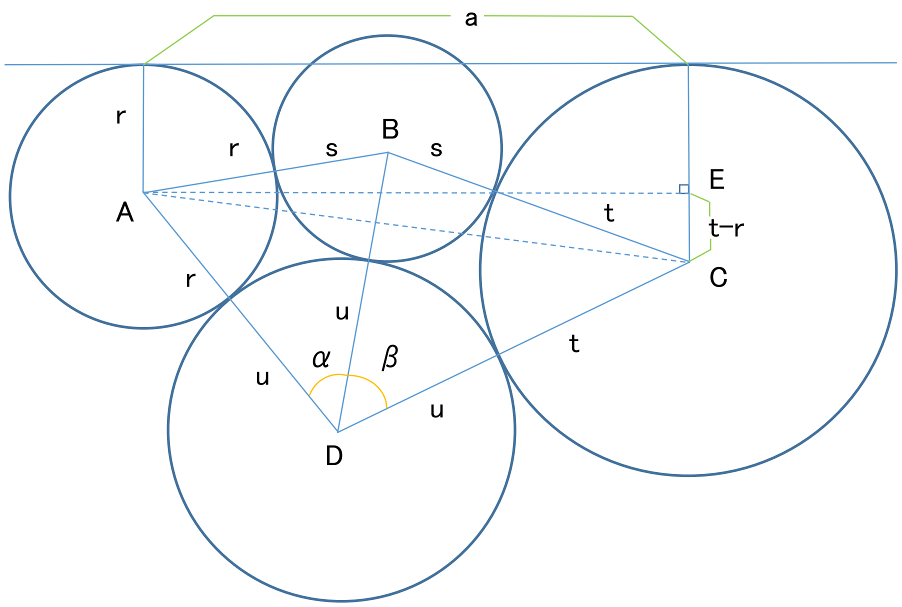
令和5年10月の問題ーNo.2

甲、乙、丙、丁の円の半径をそれぞれ r, s, t, u とし、また、子の長さを a とする。



△ADBより、 $(r+s)^2=(r+u)^2+(s+u)^2-2(r+u)(s+u)\cdot\cos(\alpha)$

$$\cos(\alpha)=1-\frac{2rs}{(r+u)(s+u)} \qquad \sin(\alpha)=\frac{2\sqrt{rsu(r+s+u)}}{(r+u)(s+u)}$$

△CDBより、 $(t+s)^2=(t+u)^2+(s+u)^2-2(t+u)(s+u)\cdot\cos(\beta)$

$$\cos(\beta)=1-\frac{2ts}{(t+u)(s+u)} \qquad \sin(\beta)=\frac{2\sqrt{stu(s+t+u)}}{(t+u)(s+u)}$$

$$\cos(\alpha+\beta)=\cos(\alpha)\cdot\cos(\beta)-\sin(\alpha)\cdot\sin(\beta)=\left[1-\frac{2rs}{(r+u)\cdot(s+u)}\right]\cdot\left[1-\frac{2ts}{(t+u)\cdot(s+u)}\right]-\frac{4su\cdot\sqrt{rt(r+s+u)\cdot(s+t+u)}}{(r+u)(t+u)(s+u)^2}$$

$$=1-\frac{2rs}{(r+u)(s+u)}-\frac{2ts}{(t+u)(s+u)}+\frac{4rts^2-4su\cdot\sqrt{rt(r+s+u)\cdot(s+t+u)}}{(r+u)(t+u)(s+u)^2}$$

$$=1-2su\cdot\frac{r(s+t+u)+t(r+s+u)+2\cdot\sqrt{rt(r+s+u)\cdot(s+t+u)}}{(r+u)\cdot(t+u)\cdot(s+u)^2}=1-2su\cdot\frac{\left(\sqrt{r(s+t+u)}+\sqrt{t(r+s+u)}\right)^2}{(r+u)\cdot(t+u)\cdot(s+u)^2}$$

△ADCより、 $AC^2=(r+u)^2+(u+t)^2-2(r+u)(u+t)\cdot\cos(\alpha+\beta)$

$$=(r+u)^2+(u+t)^2-2(r+u)(u+t)+4su(r+u)(u+t)\cdot\frac{\left(\sqrt{r(s+t+u)}+\sqrt{t(r+s+u)}\right)^2}{(r+u)\cdot(t+u)\cdot(s+u)^2}$$

$$=(t-r)^2+\frac{4su\left(\sqrt{r(s+t+u)}+\sqrt{t(r+s+u)}\right)^2}{(s+u)^2}$$

△ACEより、 $AC^2=a^2+(t-r)^2$

$$a^2+(t-r)^2=(t-r)^2+\frac{4su\left(\sqrt{r(s+t+u)}+\sqrt{t(r+s+u)}\right)^2}{(s+u)^2}$$

$a^2(s+u)^2=4su\left(\sqrt{r(s+t+u)}+\sqrt{t(r+s+u)}\right)^2$

$a^2(s+u)^2-4su(r(s+t+u)+t(r+s+u))=8su\sqrt{rt(s+t+u)(r+s+u)}$

$a^4(s+u)^4-8a^2su(s+u)^2[2rt+(r+t)\cdot(s+u)]+16s^2u^2[2rt+(r+t)\cdot(s+u)]^2=64s^2u^2rt\left[(s+u)^2+(r+t)\cdot(s+u)+rt\right]$

$a^4(s+u)^4-16a^2surt(s+u)^2-8a^2su(r+t)(s+u)^3+16s^2u^2(s+u)^2(r-t)^2=0$

$4a^4(s+u)^2-64a^2surt-32a^2su(r+t)(s+u)+64s^2u^2(r-t)^2=0$

$a^4(2s+2u)^2-4a^2 2s\cdot 2u\cdot 2r\cdot 2t-2a^2 2s\cdot 2u(2r+2t)(2s+2u)+(2s)^2(2u)^2(2r-2t)^2=0$

2r, 2s, 2t, 2u, a をそれぞれ 甲、乙、丙、丁、子に置き換えると問題文の式になる。