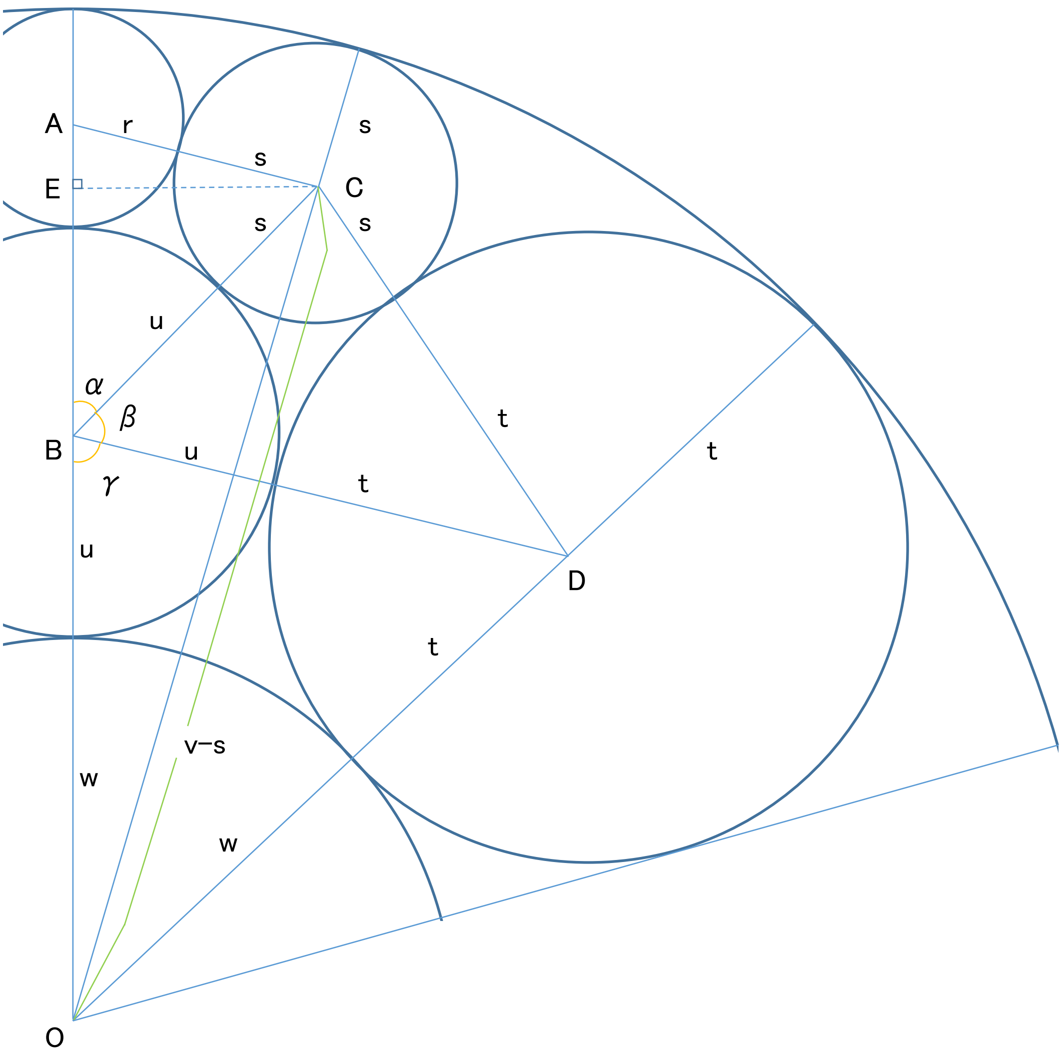
令和5年10月の問題ーNo.3

甲、乙、丙、丁の円の半径をそれぞれ r, s, t, u とする。また、扇面の外半径を v、内半径を w とする。



△ABCより、 $(r+s)^2=(r+u)^2+(s+u)^2-2(r+u)(s+u)\cdot\cos(\alpha)$

$$\cos(\alpha)=1-\frac{2rs}{(r+u)(s+u)} \qquad \sin(\alpha)=\frac{2\sqrt{rsu(r+s+u)}}{(r+u)(s+u)}$$

△BCDより、 $(s+t)^2=(s+u)^2+(u+t)^2-2(s+u)(u+t)\cdot\cos(\beta)$

$$\cos(\beta)=1-\frac{2st}{(s+u)(u+t)} \qquad \sin(\beta)=\frac{2\sqrt{stu(s+t+u)}}{(s+u)(u+t)}$$

△BODより、 $(t+w)^2=(t+u)^2+(u+w)^2-2(t+u)(u+w)\cdot\cos(\gamma)$

$$\cos(\gamma)=1-\frac{2tw}{(t+u)(u+w)}$$

$(s+u)\cdot\cos(\alpha)=a$ とおくと、△ACE、△OCEより、

$(r+s)^2-(r+u-a)^2=(v-s)^2-(u+w+a)^2$

$2(r+2u+w)a=-2rs+2ru-2vs+v^2-2uw-w^2$ 　　$w=v-2t$ より、

$2(r+2u+v-2t)a=-2rs+2ru-2vs+v^2-2u(v-2t)-(v^2-4vt+4t^2)$ 　　$t=r+u$ より、

$(v-r)\cdot a=-rs+ru-vs-uv+2ut+2vt-2t^2=u(v-r)+2r(v-r)-s(v+r)$

$a=u+2r-s\cdot\frac{v+r}{v-r}$ 　　左辺 $=(s+u)\cdot\cos(\alpha)=(s+u)-\frac{2rs}{t}$

$s+u-\frac{2rs}{t}=u+2r-s\cdot\frac{v+r}{v-r}$ 　　$s+s\cdot\frac{v+r}{v-r}=2r+\frac{2rs}{t}$ 　　$\frac{vs}{v-r}=\frac{r(t+s)}{t}$

$(r(t+s)-st)v=r^2(t+s)$

$v=\frac{r^2(t+s)}{rt-s(t-r)}=\frac{r^2(t+s)}{rt-su}$ 　　———①

$\alpha + \beta = \pi - \gamma \qquad \cos(\alpha + \beta) = \cos(\pi - \gamma) = -\cos(\gamma)$

$$\cos(\alpha+\beta) = \cos(\alpha)\cdot\cos(\beta) - \sin(\alpha)\cdot\sin(\beta) = \left[1 - \frac{2rs}{(r+u)\cdot(s+u)}\right]\cdot\left[1 - \frac{2st}{(s+u)\cdot(u+t)}\right] - \frac{4su\cdot\sqrt{rt(r+s+u)\cdot(s+t+u)}}{(r+u)(u+t)(s+u)^2}$$

$$= 1 - \frac{2rs}{(r+u)(s+u)} - \frac{2st}{(s+u)(u+t)} + \frac{4rts^2 - 4su\cdot\sqrt{rt(r+s+u)\cdot(s+t+u)}}{(r+u)(u+t)(s+u)^2}$$

$$= 1 - 2su\cdot\frac{(\sqrt{r(s+t+u)} + \sqrt{t(r+s+u)})^2}{(r+u)(u+t)(s+u)^2} = -\cos(\gamma) = -1 + \frac{2tw}{(t+u)(u+w)}$$

$$\frac{2su(\sqrt{r(s+t+u)} + \sqrt{t(r+s+u)})^2}{(r+u)(u+t)(s+u)^2} = 2 - \frac{2tw}{(t+u)(u+w)} = \frac{2u(t+u+w)}{(t+u)(u+w)}$$

$$s\cdot(\sqrt{r(s+t+u)} + \sqrt{t(r+s+u)})^2(u+w) = (r+u)(s+u)^2(t+u+w)$$

$$s\cdot(\sqrt{r(s+t+u)} + \sqrt{t(r+s+u)})^2(u+v-2t) = t(s+u)^2(u+v-t)$$

式①より、 $u + v = \dfrac{u(rt - su) + r^2(t+s)}{rt - su} = \dfrac{t(rt + s(r-u))}{rt - su}$

$$s\cdot(\sqrt{r(s+t+u)} + \sqrt{t(r+s+u)})^2 t\left(\frac{rt + s(r-u)}{rt - su} - 2\right) = t^2(s+u)^2\left(\frac{rt + s(r-u)}{rt - su} - 1\right)$$

$$s\cdot(\sqrt{r(s+t+u)} + \sqrt{t(r+s+u)})^2\cdot\left[\frac{(s-r)\cdot t^2}{rt - su}\right] = t^2\cdot(s+u)^2\cdot\left(\frac{sr}{rt - su}\right)$$

$$(\sqrt{r(s+t+u)} + \sqrt{t(r+s+u)})^2(s-r) = r(s+u)^2$$

$s + u = x$ とおくと、 $(\sqrt{r(x+t)} + \sqrt{t(x+r)})^2(s-r) = rx^2$

$$x^2\left[r^2x^2 - 2r(s-r)(r+t)x - (s-r)\left[4rst - (r+t)^2(s-r)\right]\right] = 0$$

$x \neq 0$ より、 $x = \dfrac{(s-r)\cdot(r+t) \pm 2\sqrt{rst(s-r)}}{r}$

$$u = x - s = \frac{(s-r)t \pm 2\cdot\sqrt{rst(s-r)}}{r} - r = \frac{[\sqrt{(s-r)t} \pm \sqrt{rs}]^2 - rs}{r} - r = \frac{[\sqrt{(s-r)t} \pm \sqrt{rs}]^2}{r} - (r+s)$$

$r + u + s = \dfrac{[\sqrt{(s-r)t} \pm \sqrt{rs}]^2}{r} \qquad (t+s)r = [\sqrt{(s-r)t} \pm \sqrt{rs}]^2$

$\sqrt{(t+s)r} = \sqrt{(s-r)t} \pm \sqrt{rs} \qquad \sqrt{(t+s)r} - \sqrt{(s-r)t} = \pm\sqrt{rs}$

$(t+s)r + (s-r)t - 2\sqrt{rt(t+s)(s-r)} = rs \qquad st = 2\sqrt{rt(t+s)(s-r)}$

$s^2t = 4r(s-r)(t+s) \qquad (2r-s)^2t = 4rs(s-r)$

$t = \dfrac{4rs(s-r)}{(2r-s)^2} \qquad u = t - r = \dfrac{4rs(s-r)}{(2r-s)^2} - r = \dfrac{r(3s^2 - 4r^2)}{(2r-s)^2}$

$r = \dfrac{1.75}{2} \quad s = \dfrac{2.25}{2}$ を代入すると、 $u = \dfrac{329}{200} \qquad 2u = 3.29$

∴丁円の直径 = 3.29 寸