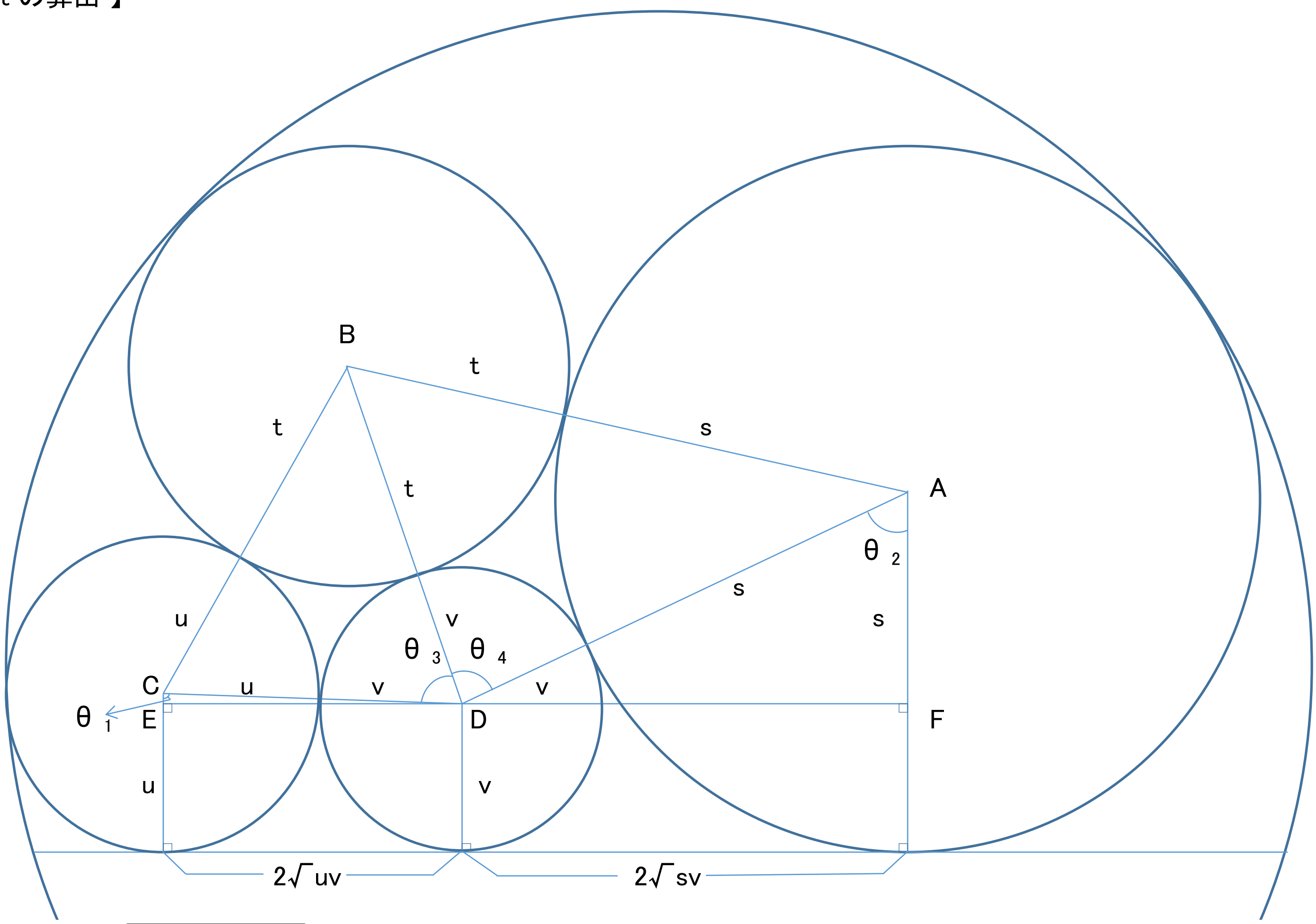
令和3年11月の問題ーNo.3

外円、甲、乙、丙、丁の円の半径をそれぞれ、r, s, t, u, v とする。

【 t の算出 】



線分ED = $\sqrt{(u+v)^2-(u-v)^2} = 2\sqrt{uv}$　　　同様に、 $DF = 2\sqrt{sv}$

△CDEより、　$\cos(\theta_1) = \dfrac{u-v}{u+v}$　　$\sin(\theta_1) = \dfrac{2\sqrt{uv}}{u+v}$

△ADFより、　$\cos(\theta_2) = \dfrac{s-v}{s+v}$　　$\sin(\theta_2) = \dfrac{2\sqrt{sv}}{s+v}$

△BDCより、　$(t+u)^2 = (t+v)^2 + (u+v)^2 - 2(t+v)(u+v)\cdot\cos(\theta_3)$　　$\cos(\theta_3) = 1 - \dfrac{2tu}{(t+v)(u+v)}$　　$\sin(\theta_3) = \dfrac{\sqrt{4tuv(t+u+v)}}{(t+v)(u+v)}$

△BDAより、　$(t+s)^2 = (t+v)^2 + (s+v)^2 - 2(t+v)(s+v)\cdot\cos(\theta_4)$　　$\cos(\theta_4) = 1 - \dfrac{2ts}{(t+v)(s+v)}$　　$\sin(\theta_4) = \dfrac{\sqrt{4tsv(t+s+v)}}{(t+v)(s+v)}$

$\left(\dfrac{\pi}{2} - \theta_1\right) + \left(\dfrac{\pi}{2} - \theta_2\right) + \theta_3 + \theta_4 = \pi$　　$\theta_3 + \theta_4 = \theta_1 + \theta_2$　　$\cos(\theta_3 + \theta_4) = \cos(\theta_1 + \theta_2)$

$$\cos(\theta_1+\theta_2) = \cos(\theta_1)\cdot\cos(\theta_2) - \sin(\theta_1)\cdot\sin(\theta_2) = \frac{u-v}{u+v}\cdot\frac{s-v}{s+v} - \frac{2\sqrt{uv}}{u+v}\cdot\frac{2\sqrt{sv}}{s+v} = \frac{(v-\sqrt{su})^2 - (\sqrt{s}+\sqrt{u})^2 v}{(u+v)(s+v)}$$

$$\cos(\theta_3+\theta_4) = \cos(\theta_3)\cdot\cos(\theta_4) - \sin(\theta_3)\cdot\sin(\theta_4) = \left[1 - \frac{2tu}{(t+v)(u+v)}\right]\left[1 - \frac{2ts}{(t+v)(s+v)}\right] - \frac{\sqrt{4tuv(t+u+v)}}{(t+v)(u+v)}\cdot\frac{\sqrt{4tsv(t+s+v)}}{(t+v)(s+v)}$$

$$= 1 - \frac{2tv(u(t+s+v) + s(t+u+v))}{(t+v)^2\cdot(s+v)\cdot(u+v)} - \frac{4tv\cdot\sqrt{us(t+u+v)\cdot(t+s+v)}}{(t+v)^2\cdot(s+v)\cdot(u+v)} = 1 - \frac{2tv(\sqrt{u(t+s+v)} + \sqrt{s(t+u+v)})^2}{(t+v)^2(u+v)(s+v)}$$

$(t+v)^2(u+v)(s+v) = 2tv(\sqrt{u(t+s+v)} + \sqrt{s(t+u+v)})^2 + (t+v)^2\left[(v-\sqrt{su})^2 - (\sqrt{s}+\sqrt{u})^2 v\right]$

$(t+v)^2\left[(u+v)(s+v) - (v-\sqrt{su})^2 + (\sqrt{s}+\sqrt{u})^2 v\right] = 2tv(\sqrt{u(t+s+v)} + \sqrt{s(t+u+v)})^2$

$2v(t+v)^2(\sqrt{s}+\sqrt{u})^2 = 2tv(\sqrt{u(t+s+v)} + \sqrt{s(t+u+v)})^2$

$(t+v)(\sqrt{s}+\sqrt{u}) = \sqrt{t}(\sqrt{u(t+s+v)} + \sqrt{s(t+u+v)})$　　　$t + v = a$　とおくと、　$t = a - v$

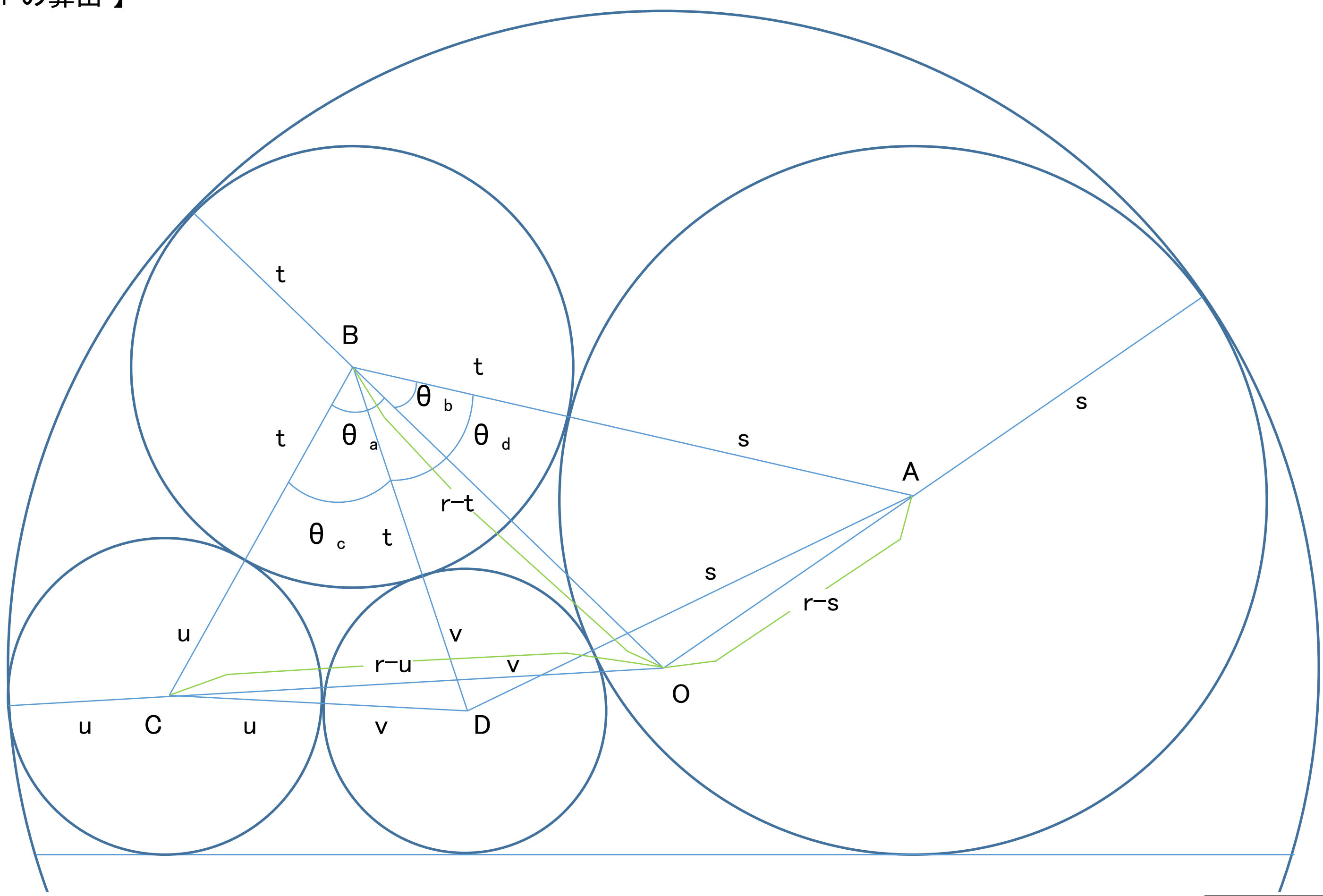
$a(\sqrt{s}+\sqrt{u}) = \sqrt{a-v}(\sqrt{s(a+u)} + \sqrt{u(a+s)})$　　　$a(\sqrt{s}+\sqrt{u})(\sqrt{s(a+u)} - \sqrt{u(a+s)}) = \sqrt{a-v}(s(a+u) - u(a+s)) = \sqrt{a-v}(s-u)a$

$\sqrt{s(a+u)} - \sqrt{u(a+s)} = \sqrt{a-v}(\sqrt{s}-\sqrt{u})$　　　$s(a+u) + u(a+s) - 2\cdot\sqrt{su(a+u)(a+s)} = (\sqrt{s}-\sqrt{u})^2(a-v)$

$4\sqrt{su}(\sqrt{s}-\sqrt{u})^2(\sqrt{su}-v)a = (\sqrt{s}-\sqrt{u})^2\left[(\sqrt{s}-\sqrt{u})^2 v^2 + 4suv\right]$

$a = \dfrac{(\sqrt{s}-\sqrt{u})^2 v + 4su}{4\sqrt{su}(\sqrt{su}-v)}v$　　　$t = a - v = \dfrac{(\sqrt{s}-\sqrt{u})^2 v + 4su - 4\sqrt{su}(\sqrt{su}-v)}{4\sqrt{su}(\sqrt{su}-v)}v = \dfrac{(\sqrt{s}+\sqrt{u})^2 v^2}{4\sqrt{su}(\sqrt{su}-v)}$

【 r の算出 】



△CBOより、 $(r-u)^2=(t+u)^2+(r-t)^2-2(t+u)(r-t)\cdot\cos(\theta_a)$ $\cos(\theta_a)=\dfrac{2ru}{(t+u)(r-t)}-1$ $\sin(\theta_a)=\dfrac{\sqrt{4rut(r-t-u)}}{(t+u)(r-t)}$

△ABOより、 $(r-s)^2=(t+s)^2+(r-t)^2-2(t+s)(r-t)\cdot\cos(\theta_b)$ $\cos(\theta_b)=\dfrac{2rs}{(t+s)(r-t)}-1$ $\sin(\theta_b)=\dfrac{\sqrt{4rst(r-t-s)}}{(t+s)(r-t)}$

△CBDより、 $(u+v)^2=(t+u)^2+(t+v)^2-2(t+u)(t+v)\cdot\cos(\theta_c)$ $\cos(\theta_c)=1-\dfrac{2uv}{(t+u)(t+v)}$ $\sin(\theta_c)=\dfrac{\sqrt{4uvt(t+v+u)}}{(t+u)(t+v)}$

△ABDより、 $(s+v)^2=(t+s)^2+(t+v)^2-2(t+s)(t+v)\cdot\cos(\theta_d)$ $\cos(\theta_d)=1-\dfrac{2sv}{(t+s)(t+v)}$ $\sin(\theta_d)=\dfrac{\sqrt{4svt(t+v+s)}}{(t+s)(t+v)}$

$\theta_a+\theta_b=\theta_c+\theta_d$ $\cos(\theta_a+\theta_b)=\cos(\theta_c+\theta_d)$

$$\cos(\theta_a+\theta_b)=\cos(\theta_a)\cdot\cos(\theta_b)-\sin(\theta_a)\cdot\sin(\theta_b)=\left[\frac{2ru}{(t+u)(r-t)}-1\right]\left[\frac{2rs}{(t+s)(r-t)}-1\right]-\frac{\sqrt{4rut(r-t-u)}}{(t+u)(r-t)}\cdot\frac{\sqrt{4rst(r-t-s)}}{(t+s)(r-t)}$$

$$=1-\frac{2rt\left(\sqrt{u(r-t-s)}+\sqrt{s(r-t-u)}\right)^2}{(t+u)(t+s)(r-t)^2}$$

$$\cos(\theta_c+\theta_d)=\cos(\theta_c)\cdot\cos(\theta_d)-\sin(\theta_c)\cdot\sin(\theta_d)=\left[1-\frac{2uv}{(t+u)(t+v)}\right]\left[1-\frac{2sv}{(t+s)(t+v)}\right]-\frac{\sqrt{4uvt(t+v+u)}}{(t+u)(t+v)}\cdot\frac{\sqrt{4svt(t+v+s)}}{(t+s)(t+v)}$$

$$=1-\frac{2vt\left(\sqrt{u(t+v+s)}+\sqrt{s(t+v+u)}\right)^2}{(t+u)(t+s)(t+v)^2}$$

$$\frac{2rt\left(\sqrt{u(r-t-s)}+\sqrt{s(r-t-u)}\right)^2}{(t+u)(t+s)(r-t)^2}=\frac{2vt\left(\sqrt{u(t+v+s)}+\sqrt{s(t+v+u)}\right)^2}{(t+u)(t+s)(t+v)^2}$$

$\sqrt{r}(t+v)\left(\sqrt{u(r-t-s)}+\sqrt{s(r-t-u)}\right)=\sqrt{v}(r-t)\left(\sqrt{u(t+v+s)}+\sqrt{s(t+v+u)}\right)$ $r-t=b$ とおくと、 $r=b+t$

$$\sqrt{b+t}(t+v)\left(\sqrt{u(b-s)}+\sqrt{s(b-u)}\right)=\sqrt{v}b\left(\sqrt{u(t+v+s)}+\sqrt{s(t+v+u)}\right)$$

$\sqrt{b+t}\left(\sqrt{u(b-s)}+\sqrt{s(b-u)}\right)=\dfrac{\sqrt{v}\left(\sqrt{u(t+v+s)}+\sqrt{s(t+v+u)}\right)}{t+v}b$ $\dfrac{\sqrt{v}\left(\sqrt{u(t+v+s)}+\sqrt{s(t+v+u)}\right)}{t+v}=c$ とおくと、

$$\sqrt{b+t}\left(\sqrt{u(b-s)}+\sqrt{s(b-u)}\right)=cb$$

$$\left[4su-\left[c^2-(s+u)\right]^2\right]b^2-2\left[c^2\left[2su-(s+u)t\right]+(s-u)^2t\right]b-t\left[4suc^2+(s-u)^2t\right]=0$$

$$b=\frac{c^2\cdot\left[2su-(s+u)\cdot t\right]+(s-u)^2\cdot t\pm 2c^2\cdot\sqrt{su\left[(t+s)\cdot(t+u)-tc^2\right]}}{4su-\left[c^2-(s+u)\right]^2}$$

$$r=b+t=\frac{c^2\cdot\left[2su+(s+u)t-tc^2\pm 2\cdot\sqrt{su\left[(t+s)\cdot(t+u)-tc^2\right]}\right]}{4su-\left[c^2-(s+u)\right]^2}=\frac{c^2\left[\left[\sqrt{su}\pm\sqrt{(t+s)(t+u)-tc^2}\right]^2-t^2\right]}{4su-\left[c^2-(s+u)\right]^2}$$

$$t+v=\frac{(\sqrt{s}+\sqrt{u})^2\cdot v^2+4\sqrt{su}\cdot(\sqrt{su}-v)v}{4\sqrt{su}(\sqrt{su}-v)}=\frac{(\sqrt{s}-\sqrt{u})^2v^2+4suv}{4\sqrt{su}(\sqrt{su}-v)}$$

$$t+v+s=\frac{(\sqrt{s}-\sqrt{u})^2\cdot v^2+4suv+4\sqrt{su}\cdot(\sqrt{su}-v)\cdot s}{4\sqrt{su}\cdot(\sqrt{su}-v)}=\frac{\left[2s\cdot\sqrt{u}-(\sqrt{s}-\sqrt{u})v\right]^2}{4\sqrt{su}\cdot(\sqrt{su}-v)}$$

$$t+v+u=\frac{(\sqrt{s}-\sqrt{u})^2\cdot v^2+4suv+4\sqrt{su}\cdot(\sqrt{su}-v)\cdot u}{4\sqrt{su}\cdot(\sqrt{su}-v)}=\frac{\left[(\sqrt{s}-\sqrt{u})v+2\cdot\sqrt{s}u\right]^2}{4\sqrt{su}\cdot(\sqrt{su}-v)}$$

$$c=\frac{\sqrt{v}(\sqrt{u(t+v+s)}+\sqrt{s(t+v+u)})}{t+v}=\frac{\sqrt{v}}{t+v}\left[\sqrt{u}\cdot\frac{2s\cdot\sqrt{u}-(\sqrt{s}-\sqrt{u})v}{\sqrt{4\sqrt{su}\cdot(\sqrt{su}-v)}}+\sqrt{s}\cdot\frac{(\sqrt{s}-\sqrt{u})v+2\cdot\sqrt{s}u}{\sqrt{4\sqrt{su}\cdot(\sqrt{su}-v)}}\right]$$

$$=\frac{1}{t+v}\cdot\frac{\sqrt{v}\cdot\left[(\sqrt{s}-\sqrt{u})^2v+4su\right]}{\sqrt{4\sqrt{su}\cdot(\sqrt{su}-v)}}=\frac{4\sqrt{su}\cdot(\sqrt{su}-v)}{\left[(\sqrt{s}-\sqrt{u})^2v+4su\right]v}\cdot\frac{\sqrt{v}\left[(\sqrt{s}-\sqrt{u})^2v+4su\right]}{\sqrt{4\sqrt{su}\cdot(\sqrt{su}-v)}}=\sqrt{\frac{4\sqrt{su}\cdot(\sqrt{su}-v)}{v}}$$

$$c^2=\frac{4\sqrt{su}\cdot(\sqrt{su}-v)}{v}=\frac{(\sqrt{s}+\sqrt{u})^2v}{t}$$ $(\sqrt{s}+\sqrt{u})^2v=d$ とおくと、 $t=\dfrac{vd}{4\sqrt{su}\cdot(\sqrt{su}-v)}$ $tc^2=d$

$$r=\frac{4\sqrt{su}\cdot(\sqrt{su}-v)v\left[\left[\sqrt{su}\pm\sqrt{(t+s)(t+u)-d}\right]^2-t^2\right]}{4suv^2-\left[4\sqrt{su}\cdot(\sqrt{su}-v)-(s+u)\cdot v\right]^2}$$

$$(t+s)(t+u)-d=t^2+(s+u)t+su-d=\left[\frac{vd}{4\sqrt{su}\cdot(\sqrt{su}-v)}\right]^2+\frac{(s+u)v-4\sqrt{su}\cdot(\sqrt{su}-v)}{4\sqrt{su}\cdot(\sqrt{su}-v)}d+su$$

$$=\frac{v^2d^2}{\left[4\sqrt{su}\cdot(\sqrt{su}-v)\right]^2}+\frac{d^2-2\sqrt{su}\cdot\left(2\sqrt{su}-v\right)\cdot d}{4\sqrt{su}\cdot(\sqrt{su}-v)}+su=\frac{\left(2\sqrt{su}-v\right)^2}{\left[4\sqrt{su}\cdot(\sqrt{su}-v)\right]^2}d^2-\frac{2\sqrt{su}\cdot\left(2\sqrt{su}-v\right)}{4\sqrt{su}\cdot(\sqrt{su}-v)}d+su$$

$$=\left[\frac{\left(2\sqrt{su}-v\right)d}{4\sqrt{su}\cdot(\sqrt{su}-v)}-\sqrt{su}\right]^2$$

rの分母 $= 4suv^2-\left[4su-4\sqrt{su}v-\left[(\sqrt{s}+\sqrt{u})^2-2\sqrt{su}\right]v\right]^2=4suv^2-\left(4su-2\sqrt{su}v-d\right)^2$

$$=\left(2\sqrt{su}\cdot v+4su-2\sqrt{su}\cdot v-d\right)\left(2\sqrt{su}\cdot v-4su+2\sqrt{su}\cdot v+d\right)=\left(4su-d\right)\left[d-4\sqrt{su}\cdot(\sqrt{su}-v)\right]$$

(i) $$r=\frac{4\sqrt{su}\cdot(\sqrt{su}-v)v\cdot\left[\left[\sqrt{su}+\frac{\left(2\sqrt{su}-v\right)d}{4\sqrt{su}\cdot(\sqrt{su}-v)}-\sqrt{su}\right]^2-\left[\frac{vd}{4\sqrt{su}\cdot(\sqrt{su}-v)}\right]^2\right]}{\left(4su-d\right)\left[d-4\sqrt{su}\cdot(\sqrt{su}-v)\right]}$$

分子 $= 4\sqrt{su}\cdot(\sqrt{su}-v)\cdot v\cdot\left[\left[\frac{\left(2\sqrt{su}-v\right)\cdot d}{4\sqrt{su}\cdot(\sqrt{su}-v)}\right]^2-\left[\frac{vd}{4\sqrt{su}\cdot(\sqrt{su}-v)}\right]^2\right]=\frac{\left(2\sqrt{su}-v\right)^2-v^2}{4\sqrt{su}\cdot(\sqrt{su}-v)}vd^2=vd^2$

$$r=\frac{vd^2}{\left(4su-d\right)\left[d-4\cdot\sqrt{su}\cdot(\sqrt{su}-v)\right]}=\frac{vd^2}{\left(4su-d\right)\left[4\sqrt{su}v-\left(4su-d\right)\right]}$$

(ii) $$r=\frac{4\sqrt{su}\cdot(\sqrt{su}-v)v\cdot\left[\left[\sqrt{su}-\frac{\left(2\sqrt{su}-v\right)d}{4\sqrt{su}\cdot(\sqrt{su}-v)}+\sqrt{su}\right]^2-\left[\frac{vd}{4\sqrt{su}\cdot(\sqrt{su}-v)}\right]^2\right]}{\left(4su-d\right)\left[d-4\sqrt{su}\cdot(\sqrt{su}-v)\right]}$$

分子 $= 4\sqrt{su}\cdot(\sqrt{su}-v)v\cdot\left[\left[2\sqrt{su}-\frac{\left(2\sqrt{su}-v\right)\cdot d}{4\sqrt{su}\cdot(\sqrt{su}-v)}\right]^2-\left[\frac{vd}{4\sqrt{su}\cdot(\sqrt{su}-v)}\right]^2\right]$

$$=4\sqrt{su}\cdot(\sqrt{su}-v)v\cdot\left[2\sqrt{su}-\frac{\left(2\sqrt{su}-v\right)\cdot d}{4\sqrt{su}\cdot(\sqrt{su}-v)}+\frac{vd}{4\sqrt{su}\cdot(\sqrt{su}-v)}\right]\cdot\left[2\sqrt{su}-\frac{\left(2\sqrt{su}-v\right)\cdot d}{4\sqrt{su}\cdot(\sqrt{su}-v)}-\frac{vd}{4\sqrt{su}\cdot(\sqrt{su}-v)}\right]$$

$$=4\sqrt{su}\cdot(\sqrt{su}-v)v\cdot\left[2\sqrt{su}-\frac{2\left(\sqrt{su}-v\right)\cdot d}{4\sqrt{su}\cdot(\sqrt{su}-v)}\right]\cdot\left[2\sqrt{su}-\frac{2\sqrt{su}\cdot d}{4\sqrt{su}\cdot(\sqrt{su}-v)}\right]$$

$$=4\sqrt{su}\cdot(\sqrt{su}-v)\cdot v\cdot\left(2\sqrt{su}-\frac{d}{2\sqrt{su}}\right)\cdot\left(2\sqrt{su}\right)\cdot\left[\frac{4\sqrt{su}\cdot(\sqrt{su}-v)-d}{4\sqrt{su}\cdot(\sqrt{su}-v)}\right]=v\left(4su-d\right)\cdot\left[4\sqrt{su}\cdot(\sqrt{su}-v)-d\right]$$

$$r=\frac{v\left(4su-d\right)\left[4\sqrt{su}\cdot(\sqrt{su}-v)-d\right]}{\left(4su-d\right)\left[d-4\sqrt{su}\cdot(\sqrt{su}-v)\right]}=-v$$ <0 なので除外する。

$$r=\frac{vd^2}{\left(4su-d\right)\left[4\sqrt{su}v-\left(4su-d\right)\right]}$$ 但し、 $d=(\sqrt{s}+\sqrt{u})^2v$

$s=\dfrac{135}{2}$ $u=\dfrac{60}{2}$ $v=\dfrac{54}{2}$ を代入すると、 $r=125$ $2r=250$

∴ 外円径 = 250寸