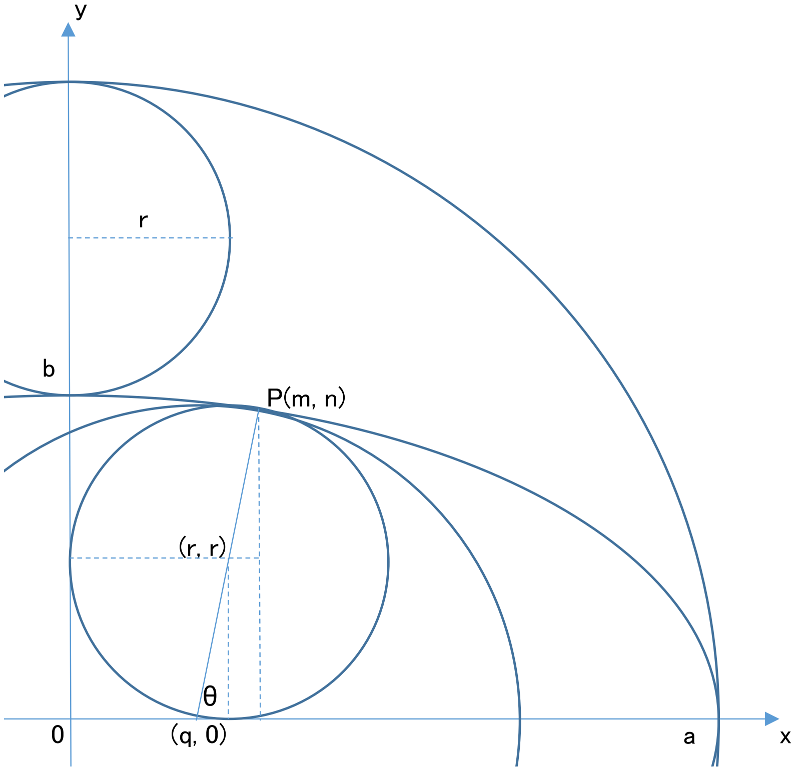
令和4年6月の問題ーNo.2



楕円の長軸の長さを 2a、短軸の長さを 2bとし、等円、大円の半径をそれぞれ r、aとする。

等円と楕円の接点をP(m, n)とし、図のように点Pで楕円に接し、中心座標(q, 0)で半径sの円を追加する。
この円と楕円が接するので判別式=0により、点Pのx座標mを求める。

$$(x-q)^2+y^2=s^2$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad y^2=b^2-\frac{b^2}{a^2}x^2$$

$$(x-q)^2+\left(b^2-\frac{b^2}{a^2}x^2\right)=s^2 \qquad \frac{a^2-b^2}{a^2}x^2-2qx+\left(q^2+b^2-s^2\right)=0$$

$$x=\frac{q}{\left(\frac{a^2-b^2}{a^2}\right)}=\frac{a^2}{a^2-b^2}q \qquad m=\frac{a^2}{a^2-b^2}q$$

$$m=r+r\cdot\cos(\theta)=r(1+\cos(\theta))$$

$$n=r+r\cdot\sin(\theta)=r(1+\sin(\theta))$$

$$n=s\cdot\sin(\theta)=r(1+\sin(\theta)) \qquad s=\frac{1+\sin(\theta)}{\sin(\theta)}r$$

$$q=m-s\cdot\cos(\theta)=r(1+\cos(\theta))-\frac{1+\sin(\theta)}{\sin(\theta)}\cdot r\cdot\cos(\theta)=\frac{(1+\cos(\theta))\cdot\sin(\theta)-(1+\sin(\theta))\cdot\cos(\theta)}{\sin(\theta)}r$$

$$q=\frac{\sin(\theta)-\cos(\theta)}{\sin(\theta)}r$$

$$m=r(1+\cos(\theta))=\frac{a^2}{a^2-b^2}\cdot q=\frac{a^2}{a^2-b^2}\cdot\frac{\sin(\theta)-\cos(\theta)}{\sin(\theta)}r$$

$$1+\cos(\theta)=\frac{a^2}{a^2-b^2}\cdot\frac{\sin(\theta)-\cos(\theta)}{\sin(\theta)} \qquad \left(a^2-b^2\right)\cdot\sin(\theta)(1+\cos(\theta))=a^2(\sin(\theta)-\cos(\theta))$$

$$a^2\cdot\cos(\theta)(1+\sin(\theta))=b^2\cdot\sin(\theta)(1+\cos(\theta))$$

楕円はP(r(1+cosθ), r(1+sinθ))を通るので、

$$b^2r^2(1+\cos(\theta))^2+a^2r^2(1+\sin(\theta))^2=a^2b^2$$

$$a^2\cdot\frac{\cos(\theta)\cdot(1+\sin(\theta))}{\sin(\theta)(1+\cos(\theta))}r^2(1+\cos(\theta))^2+a^2r^2(1+\sin(\theta))^2=a^2b^2$$

$$b^2=\left[\frac{\cos(\theta)}{\sin(\theta)}(1+\sin(\theta))(1+\cos(\theta))+(1+\sin(\theta))^2\right]\cdot r^2=\frac{1+\sin(\theta)}{\sin(\theta)}\left[\cos(\theta)(1+\cos(\theta))+\sin(\theta)(1+\sin(\theta))\right]r^2$$

$$b^2=\frac{(1+\sin(\theta))(1+\sin(\theta)+\cos(\theta))}{\sin(\theta)}r^2$$

$$a^2=b^2\cdot\frac{\sin(\theta)\cdot(1+\cos(\theta))}{\cos(\theta)\cdot(1+\sin(\theta))}=\frac{(1+\sin(\theta))\cdot(1+\sin(\theta)+\cos(\theta))}{\sin(\theta)}\cdot r^2\cdot\frac{\sin(\theta)\cdot(1+\cos(\theta))}{\cos(\theta)(1+\sin(\theta))}$$

$$a^2=\frac{(1+\cos(\theta))\cdot(1+\sin(\theta)+\cos(\theta))}{\cos(\theta)}\cdot r^2$$

$b = a - 2r \qquad a - b = 2r$

$$\sqrt{1+\sin(\theta)+\cos(\theta)}\left(\sqrt{\frac{1+\cos(\theta)}{\cos(\theta)}}-\sqrt{\frac{1+\sin(\theta)}{\sin(\theta)}}\right)r = 2r$$

$$(1+\sin(\theta)+\cos(\theta))\left[\frac{1+\cos(\theta)}{\cos(\theta)}+\frac{(1+\sin(\theta))}{\sin(\theta)}-2\sqrt{\frac{(1+\sin(\theta))(1+\cos(\theta))}{\sin(\theta)\cdot\cos(\theta)}}\right]=4$$

$$(1+\sin(\theta)+\cos(\theta))\left[\sin(\theta)(1+\cos(\theta))+\cos(\theta)(1+\sin(\theta))-2\sqrt{(\sin(\theta))\cdot\cos(\theta)\cdot(1+\sin(\theta))(1+\cos(\theta))}\right]=4\cdot\sin(\theta)\cdot\cos(\theta)$$

$$(1+\sin(\theta)+\cos(\theta))\left[(\sin(\theta)+\cos(\theta))+2\cdot\sin(\theta)\cdot\cos(\theta)-2\sqrt{\sin(\theta)\cdot\cos(\theta)\cdot\left[1+(\sin(\theta)+\cos(\theta))+\sin(\theta)\cdot\cos(\theta)\right]}\right]=4\cdot\sin(\theta)\cdot\cos(\theta)$$

$\sin(\theta)+\cos(\theta)=t$ とおくと、

$t^2=(\sin(\theta)+\cos(\theta))^2=1+2\cdot\sin(\theta)\cdot\cos(\theta) \qquad 2\cdot\sin(\theta)\cdot\cos(\theta)=t^2-1$

$$(1+t)\left[t+(t^2-1)-2\sqrt{\frac{t^2-1}{2}\left(1+t+\frac{t^2-1}{2}\right)}\right]=2(t^2-1)$$

$$t^2+t-1-\sqrt{(t^2-1)(t^2-1+2t+2)}=2(t-1)$$

$$\sqrt{(t^2-1)(t^2+2t+1)}=t^2-t+1$$

$$(t^2-1)(t^2+2t+1)=(t^2-t+1)^2$$

$$4t^3-3t^2-2=0 \qquad t^3-\frac{3}{4}t^2-\frac{1}{2}=0$$

$t=u+\frac{1}{4}$ とおくと、 $u^3+\frac{3}{4}u^2+\frac{3}{16}u+\frac{1}{64}-\frac{3}{4}\left(u^2+\frac{1}{2}u+\frac{1}{16}\right)-\frac{1}{2}=0$

$$u^3-\frac{3}{16}u-\frac{17}{32}=0 \qquad p=\frac{-3}{16} \qquad q=\frac{-17}{32}$$

$$\frac{-q}{2}+\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}=\frac{17}{64}+\sqrt{\frac{17^2}{64^2}-\frac{1}{16^3}}=\frac{17+12\sqrt{2}}{64}$$

$$\frac{-q}{2}-\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}=\frac{17-12\sqrt{2}}{64}$$

$$u=\sqrt[3]{\frac{17+12\sqrt{2}}{64}}+\sqrt[3]{\frac{17-12\sqrt{2}}{64}}=\frac{\sqrt[3]{17+12\sqrt{2}}+\sqrt[3]{17-12\sqrt{2}}}{4}$$

$$t=u+\frac{1}{4}=\frac{\sqrt[3]{17+12\sqrt{2}}+\sqrt[3]{17-12\sqrt{2}}+1}{4}=\frac{\sqrt[3]{(3+2\sqrt{2})^2}+\sqrt[3]{(3-2\sqrt{2})^2}+1}{4}$$

$$t=\frac{\sqrt[3]{(3+2\cdot\sqrt{2})^2}+\sqrt[3]{(3-2\cdot\sqrt{2})^2}+\sqrt[3]{(3+2\cdot\sqrt{2})\cdot(3-2\cdot\sqrt{2})}}{4}\cdot\frac{\sqrt[3]{3+2\cdot\sqrt{2}}-\sqrt[3]{3-2\cdot\sqrt{2}}}{\sqrt[3]{3+2\sqrt{2}}-\sqrt[3]{3-2\sqrt{2}}}=\frac{\sqrt[3]{(3+2\sqrt{2})^3}-\sqrt[3]{(3-2\sqrt{2})^3}}{4\left(\sqrt[3]{3+2\sqrt{2}}-\sqrt[3]{3-2\sqrt{2}}\right)}$$

$$t=\frac{(3+2\sqrt{2})-(3-2\sqrt{2})}{4\left(\sqrt[3]{3+2\sqrt{2}}-\sqrt[3]{3-2\sqrt{2}}\right)}=\frac{\sqrt{2}}{\sqrt[3]{3+2\sqrt{2}}-\sqrt[3]{3-2\sqrt{2}}}$$

$\sin(\theta)+\cos(\theta)=t \qquad \sin(\theta)\cdot\cos(\theta)=\frac{t^2-1}{2}$ より、

$x^2-tx+\frac{t^2-1}{2}=0 \qquad 2x^2-2tx+(t^2-1)=0$

x が sinθ 、cosθ となる。図よりθ＞45° なので sinθ＞cosθ となり、

$\sin(\theta)=\frac{t+\sqrt{2-t^2}}{2} \qquad \cos(\theta)=\frac{t-\sqrt{2-t^2}}{2}$

$$a^2=\left(1+\frac{1}{\cos(\theta)}\right)(1+\sin(\theta)+\cos(\theta))r^2=\left(1+\frac{2}{t-\sqrt{2-t^2}}\right)(1+t)r^2$$

$$a^2=\left[1+\frac{2\left(t+\sqrt{2-t^2}\right)}{t^2-(2-t^2)}\right](1+t)r^2=\left(1+\frac{t+\sqrt{2-t^2}}{t^2-1}\right)(1+t)r^2=\left(t+1+\frac{t+\sqrt{2-t^2}}{t-1}\right)r^2$$

$$\sqrt{2-t^2}=\sqrt{2+(3t^2-4t^2)}=\sqrt{(3t^2+2)-4t^2}=\sqrt{4t^3-4t^2}=\sqrt{4t^2(t-1)}=2t\cdot\sqrt{t-1}$$

$$a^2=\left(t+1+\frac{t+2t\cdot\sqrt{t-1}}{t-1}\right)r^2=\left(1+t+\frac{t}{t-1}+\frac{2t\cdot\sqrt{t-1}}{t-1}\right)r^2=\left(1+\frac{t^2}{t-1}+\frac{2t}{\sqrt{t-1}}\right)r^2=\left(1+\frac{t}{\sqrt{t-1}}\right)^2r^2$$

$$a=\left(1+\frac{t}{\sqrt{t-1}}\right)r$$

等円径が１寸なので 2r=1

大円径 $2a=\left(1+\frac{t}{\sqrt{t-1}}\right)\cdot 2r=1+\frac{t}{\sqrt{t-1}}=1+\frac{\frac{\sqrt{2}}{\sqrt[3]{3+2\sqrt{2}}-\sqrt[3]{3-2\sqrt{2}}}}{\sqrt{\frac{\sqrt{2}}{\sqrt[3]{3+2\sqrt{2}}-\sqrt[3]{3-2\sqrt{2}}}-1}}$

$$2a=1+\sqrt{\frac{2}{\left(\sqrt[3]{3+2\sqrt{2}}-\sqrt[3]{3-2\sqrt{2}}\right)\cdot\left(\sqrt{2}-\sqrt[3]{3+2\sqrt{2}}+\sqrt[3]{3-2\sqrt{2}}\right)}} \quad ≒ 4.073 \text{ 寸}$$